

*Dedicated to the Memory of  
Anatoly Alexandrovich Vlasov<sup>1</sup>*

# Oscillations of Degenerate Plasma in Layer with Specular – Accommodative Boundary Conditions

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### Abstract

In the present paper the linearized problem of plasma oscillations in layer (particularly, in thin films) in external longitudinal alternating electric field is solved analytically. Specular – accommodative boundary conditions of electron reflection from the plasma boundary are considered. Coefficients of continuous and discrete spectra of the problem are found, and electron distribution function on the plasma boundary and electric field are expressed in explicit form. Absorption of energy of electric field in layer is calculated.

Refs. 34. Figs. 2.

**Keywords:** degenerate plasma, layer, specular–accommodative boundary condition, plasma mode, expansion by eigen functions, singular integral equation, absorption energy of electric field.

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## 1 Introduction

The present paper is devoted to degenerate electron plasma behaviour research. Analysis of processes taking place in plasma under effect of external electric field, plasma waves oscillations with various types of conditions of electron reflection from the boundary has important significance today in connection with problems of such intensively developing fields as microelectronics and nanotechnologies [1] – [6].

The concept of "plasma" appeared in the works of Tonks and Langmuir for the first time (see [7]–[9]), the concept of "plasma frequency" was introduced in the same works and first questions of plasma oscillations were considered there. However, in these works equation for the electric field was considered separately from the kinetic equation.

A.A. Vlasov [10] for the first time introduced the concept of "self-consistent electric field" and added the corresponding item to the kinetic equation. Now equations describing plasma behaviour consist of anchor system of equations of Maxwell and Boltzmann. The problem of electron plasma oscillations was considered by A.A. Vlasov [10] by means of solution of the kinetic equation which included self-consistent electric field.

L.D. Landau [11] had supposed that outside of the half-space containing degenerate plasma external electromagnetic field causing oscillations in plasma is situated. By this Landau has formulated a boundary condition on the plasma boundary. After that the

problem of plasma oscillation turns out to be formulated correctly as a boundary value problem of mathematical physics.

In [11] L.D. Landau has solved analytically by Fourier series the problem of collisionless plasma behaviour in a half-space, situated in external longitudinal (perpendicular to the surface) electric field, in conditions of specular reflection of electrons from the boundary.

Further the problem of electron plasma oscillations was considered by many authors. Full analytical solution of the problem is given in the works [12] and [13].

This problem has important significance in the theory of plasma (see, for instance, [2], [14] and the references in these works, and also [15], [16]).

The problem of plasma oscillations with diffuse boundary condition was considered in the works [17], [18] by method of integral transformations. In the works [19], [20] general asymptotic analysis of electric field behaviour at the large distance from the surface was carried out. In the work [19] particular significance of plasma behaviour analysis close to plasma resonance was shown. And in the same work [19] it was stated that plasma behaviour in this case for conditions of specular and diffuse electron scattering on the surface differs substantially.

In the works [22] and [22] general questions of this problem solvability were considered, but diffuse boundary conditions were taken into account. In the work [22] structure of discrete spectrum in dependence of parameters of the problem was analyzed. The detailed analysis of the solution in general case in the works mentioned above hasn't been carried out considering the complex character of this solution.

The present work is a continuation of electron plasma behaviour in external longitudinal alternating electric field research [22] – [26].

In the present paper the linearized problem of plasma oscillations in external alternating electric field in layer (particularly, in thin films) is solved analytically. Specular – accommodative boundary conditions for electron reflection from the boundary are considered. In [24]–[26] diffuse boundary conditions were considered.

The coefficients of continuous and discrete spectra of the problem are obtained in the present work, which allows us to derive expressions for electron distribution function at the boundary of conductive medium and electric field in explicit form, to reveal the dependence of this expressions on normal momentum accommodation coefficient and to show that in the case when normal electron momentum accommodation coefficient equals to zero electron distribution function and electric field are expressed by known formulas obtained earlier in [12], [13].

The present work is a continuation of our work [27], in which questions of plasma waves specular reflection from the plane boundary bounding degenerate plasma were considered.

Let us note, that questions of plasma oscillations are also considered in nonlinear

statement (see, for instance, the work [28], [29]).

## 2 Formulation of problem

Let degenerate plasma occupy a slab (particularly, thin films)  $-a < x < a$ .

We take system of equations describing plasma behaviour. As a kinetic equation we take Boltzmann — Vlasov  $\tau$ -model kinetic equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + e \mathbf{E} \frac{\partial f}{\partial \mathbf{p}} = \frac{f_{eq}(\mathbf{r}, t) - f(\mathbf{r}, \mathbf{v}, t)}{\tau}. \quad (1.1)$$

Here  $f = f(\mathbf{r}, \mathbf{v}, t)$  is the electron distribution function,  $e$  is the electron charge,  $\mathbf{p} = m\mathbf{v}$  is the momentum of an electron,  $m$  is the electron mass,  $\tau$  is the character time between two collisions,  $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$  is the self-consistent electric field inside plasma,  $f_{eq} = f_{eq}(\mathbf{r}, t)$  is the local equilibrium Fermi — Dirac distribution function,  $f_{eq} = \Theta(\mathcal{E}_F(t, x) - \mathcal{E})$ , where  $\Theta(x)$  is the function of Heaviside,

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$$

$\mathcal{E}_F(t, x) = \frac{1}{2}mv_F^2(t, x)$  is the disturbed kinetic energy of Fermi,  $\mathcal{E} = \frac{1}{2}mv^2$  is the kinetic energy of electron.

Let us take the Maxwell equation for electric field

$$\text{div } \mathbf{E}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t). \quad (1.2)$$

Here  $\rho(\mathbf{r}, t)$  is the charge density,

$$\rho(\mathbf{r}, t) = e \int (f(\mathbf{r}, \mathbf{v}, t) - f_0(\mathbf{v})) d\Omega_F, \quad (1.3)$$

where

$$d\Omega_F = \frac{2d^3p}{(2\pi\hbar)^3}, \quad d^3p = dp_x dp_y dp_z.$$

Here  $f_0$  is the undisturbed Fermi — Dirac electron distribution function,

$$f_0(\mathcal{E}) = \Theta(\mathcal{E}_F - \mathcal{E}),$$

$\hbar$  is the Planck's constant,  $\nu$  is the effective frequency of electron collisions,  $\nu = 1/\tau$ ,  $\mathcal{E}_F = \frac{1}{2}mv_F^2$  is the undisturbed kinetic energy of Fermi,  $v_F$  is the electron velocity at the Fermi surface, which is supposed to be spherical.

We assume that external electric field outside the plasma is perpendicular to the plasma boundary and changes according to the following law:  $E_0 \exp(-i\omega t)$ .

Then one can consider that self-consistent electric field  $\mathbf{E}(\mathbf{r}, t)$  inside plasma has one  $x$ -component and changes only lengthwise the axis  $x$ :

$$\mathbf{E} = \{E_x(x, t), 0, 0\}.$$

Under this configuration the electric field is perpendicular to the boundary of plasma, which is situated in the plane  $x = 0$ .

We will linearize the local equilibrium Fermi — Dirac distribution  $f_{eq}$  in regard to the undisturbed distribution  $f_0(\mathcal{E})$ :

$$f_{eq} = f_0(\mathcal{E}) + [\mathcal{E}_F(x, t) - \mathcal{E}] \delta(\mathcal{E}_F - \mathcal{E}),$$

where  $\delta(x)$  is the delta — function of Dirac.

We also linearize the electron distribution function  $f$  in terms of absolute Fermi — Dirac distribution  $f_0(\mathcal{E})$ :

$$f = f_0(\mathcal{E}) + f_1(x, \mathbf{v}, t). \quad (1.4)$$

After the linearization of the equations (1.1)–(1.3) with the help of (1.4) we obtain the following system of equations:

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} + \nu f_1(x, \mathbf{v}, t) = \delta(\mathcal{E}_F - \mathcal{E}) [e E_x(x, t) v_x + \nu [\mathcal{E}_F(x, t) - \mathcal{E}_F]], \quad (1.5)$$

$$\frac{\partial E_x(x, t)}{\partial x} = \frac{8\pi e}{(2\pi\hbar)^3} \int f_1(x, \mathbf{v}', t) d^3 p' \quad (1.6)$$

From the law of preservation of number of particles

$$\int f_{eq} d\Omega_F = \int f d\Omega_F$$

we find:

$$[\mathcal{E}_F(x, t) - \mathcal{E}_F] \int \delta(\mathcal{E}_F - \mathcal{E}) d^3 p = \int f_1 d^3 p. \quad (1.7)$$

From the equation (1.5) it is seen that we should search for the function  $f_1$  in the form proportional to the delta — function:

$$f_1 = \delta(\mathcal{E}_F - \mathcal{E}) H(x, \mu, t), \quad \mu = \frac{v_x}{v}. \quad (1.8)$$

The system of equations (1.5) and (1.6) with the help of (1.7) and (1.8) can be transformed to the following form:

$$\frac{\partial H}{\partial t} + v_F \mu \frac{\partial H}{\partial x} + \nu H(x, \mu, t) = e v_F \mu E_x(x, t) + \frac{\nu}{2} \int_{-1}^1 H(x, \mu', t) d\mu',$$

$$\frac{\partial E_x(x, t)}{\partial x} = \frac{16\pi^2 em^2 v_F}{(2\pi\hbar)^3} \int_{-1}^1 H(x, \mu', t) d\mu'.$$

Further we introduce dimensionless functions

$$e(x_1, t) = \frac{E_x(x, t)}{E_0}, \quad h(x_1, \mu, t) = \frac{H(x, \mu, t)}{eaE_0},$$

and pass to dimensionless coordinate  $x_1 = x/a$ . We obtain the following system of equations

$$\frac{\partial H}{\partial t_1} + \mu \frac{\partial H}{\partial x_1} + \nu H(x_1, \mu, t_1) = \mu e(x_1, t_1) + \frac{1}{2} \int_{-1}^1 H(x_1, \mu', t_1) d\mu', \quad (1.9)$$

$$\frac{\partial e(x_1, t_1)}{\partial x_1} = \frac{16\pi^2 e^2 m^2 v_F a^2}{(2\pi\hbar)^3} \int_{-1}^1 H(x_1, \mu', t_1) d\mu'. \quad (1.10)$$

Here  $\omega_p$  is the electron (Langmuir) frequency of plasma oscillations,

$$\omega_p^2 = \frac{4\pi e^2 N}{m},$$

$N$  is the numerical density (concentration),  $m$  is the electron mass.

We used the following well-known relation for degenerate plasma for the conclusion of the equations (1.9) and (1.10)

$$\left(\frac{v_F m}{\hbar}\right)^3 = 3\pi^2 N.$$

### 3 Boundary conditions statement

Let us outline the time variable of the functions  $H(x_1, \mu, t_1)$  and  $e(x_1, t_1)$ , assuming

$$H(x_1, \mu, t_1) = e^{-i\omega_1 t_1} h(x_1, \mu), \quad (2.1)$$

$$e(x_1, t_1) = e^{-i\omega_1 t_1} e(x_1). \quad (2.2)$$

The system of equations (1.9) and (1.10) in this case will be transformed to the following form:

$$\mu \frac{\partial h}{\partial x_1} + w_0 h(x_1, \mu) = \mu e(x_1) + \frac{y_0}{2} \int_{-1}^1 h(x_1, \mu') d\mu', \quad (2.3)$$

$$\frac{de(x_1)}{dx_1} = \frac{u_p^2}{2} \int_{-1}^1 h(x_1, \mu') d\mu'. \quad (2.4)$$

Here

$$w_0 = y_0 - ix_0 = \frac{a}{v_F}(\nu - i\omega), \quad y_0 = \frac{a\nu}{v_F}, \quad x_0 = \frac{a\omega}{v_F}, \quad u_p^2 = 3\left(\frac{a\omega_p}{v_F}\right)^2.$$

The constant  $u_p$  can be expressed through Debaye radius  $r_D$

$$u_p^2 = 3\frac{a^2}{r_D^2}, \quad r_D = \frac{v_F}{\omega_p}.$$

Further instead of  $x_1$  we write  $x$ . We rewrite the system of equations (2.3) and (2.4) in the form:

$$\mu \frac{\partial h}{\partial x} + w_0 h(x, \mu) = \mu e(x) + \frac{y_0}{2} \int_{-1}^1 h(x, \mu') d\mu'. \quad (2.5)$$

$$\frac{de(x)}{dx} = \frac{u_p^2}{2} \int_{-1}^1 h(x, \mu') d\mu'. \quad (2.6)$$

For electric field in plasma on its border the boundary condition is satisfied

$$e(-1) = e_s, \quad e(+1) = e_s, \quad (2.7)$$

where 1 is the dimensionless depth (width) of a semilayer.

Condition of symmetry of boundary conditions (2.7) and the equations (2.5) and (2.6) mean, that electric field  $e(x)$  in the layer possess properties of symmetry

$$e(x) = e(-x). \quad (2.7a)$$

The non-flowing condition for the particle (electric current) flow through the plasma boundary means that

$$\int_{-1}^1 \mu h(-1, \mu) d\mu = \int_{-1}^1 \mu h(1, \mu) d\mu = 0. \quad (2.8)$$

In the kinetic theory for the description of the surface properties the accommodation coefficients are used often. Tangential momentum and energy accommodation coefficients are the most-used. For the problem considered the normal electron momentum accommodation under the scattering on the surface has the most important significance.

Owing to properties of symmetry of electric field and distribution function concerning a plane — middle of a layer — further let's enter the coefficient of accommodation of a normal momentum through momentum of electron streams on the bottom surface of the layer.

The normal momentum accommodation coefficient is defined by the following relation

$$\alpha_p = \frac{P_i - P_r}{P_i - P_s}, \quad 0 \leq \alpha_p \leq 1, \quad (2.9)$$

where  $P_i$  and  $P_r$  are the flows of normal to the surface momentum of incoming to the boundary and reflected from it electrons,

$$P_i = \int_{-1}^0 \mu^2 h(-1, \mu) d\mu, \quad (2.10)$$

$$P_r = \int_0^1 \mu^2 h(-1, \mu) d\mu, \quad (2.11)$$

quantity  $P_s$  is the normal momentum flow for electrons reflected from the surface which are in thermodynamic equilibrium with the wall,

$$P_s = \int_0^1 \mu^2 h_s(\mu) d\mu, \quad (2.12)$$

where the function

$$h_s(\mu) = A_s, \quad 0 < \mu < 1,$$

is the equilibrium distribution function of the corresponding electrons. This function is to satisfy the condition similar to the non-flowing condition

$$\int_{-1}^0 \mu h(-1, \mu) d\mu + \int_0^1 \mu h_s(\mu) d\mu = 0. \quad (2.13)$$

We are going to consider the relation between the normal momentum accommodation coefficient  $\alpha_p$  and the diffuseness coefficient  $q$  for the case of specular and diffuse boundary conditions which are written in the following form

$$h(-1, \mu) = (1 - q)h(-1, -\mu) + a_s, \quad 0 < \mu < 1. \quad (2.14)$$

Here  $q$  is the diffusivity coefficient ( $0 \leq q \leq 1$ ),  $a_s$  is the quantity determined from the non-flowing condition.

From the non-flowing condition we derive

$$\int_{-1}^1 \mu h(-1, \mu) d\mu = \int_{-1}^0 \mu h(-1, \mu) d\mu + \int_0^1 \mu h(-1, \mu) d\mu = 0.$$



In the second integral we replace the integrand according to the right-hand side of the specular–diffuse boundary condition (2.14). After that, using the obvious change of integration variable, we obtain that

$$a_s = -2q \int_{-1}^0 \mu h(-1, \mu) d\mu.$$

Let us use the boundary condition (2.13). Using the analogous to the preceded line of reasoning we get

$$A_s = -2 \int_{-1}^0 \mu h(-1, \mu) d\mu.$$

From the two last equations we find that

$$a_s = qA_s. \quad (2.15)$$

Further we find the difference between two flows

$$P_i - P_r = \int_{-1}^0 \mu^2 h(-1, \mu) d\mu - \int_0^1 \mu^2 h(-1, \mu) d\mu.$$

In the second integral we use the boundary condition (2.14) again. With the help of (2.15) we obtain that

$$\begin{aligned} P_i - P_r &= q \int_{-1}^0 \mu^2 h(-1, \mu) d\mu - \int_0^1 \mu^2 a_s d\mu = \\ &= q \int_{-1}^0 \mu^2 h(-1, \mu) d\mu - q \int_0^1 \mu^2 A_s d\mu = qP_i - qP_s. \end{aligned}$$

Substituting the expressions obtained to the definition of the normal momentum accommodation coefficient, we have

$$\alpha_p = \frac{P_i - P_r}{P_i - P_s} = \frac{qP_i - qP_s}{P_i - P_s} = q.$$

Thus, for specular – diffuse boundary conditions normal momentum accommodation coefficient  $\alpha_p$  coincides with the diffuseness coefficient  $q$ .

Equally with the specular – diffuse boundary conditions another variants of boundary conditions are used in kinetic theory as well.

In particular, accommodation boundary conditions are used widely. They are divided into two types: diffuse – accommodative and specular – accommodative boundary conditions (see [30]).

We consider specular – accommodative boundary conditions. For the function  $h$  this conditions will be written in the following form

$$h(-1, \mu) = h(-1, -\mu) + A_0 + A_1\mu, \quad 0 < \mu < 1. \quad (2.16)$$

$$h(1, \mu) = h(1, -\mu) - A_0 + A_1\mu, \quad -1 < \mu < 0. \quad (2.17)$$

Let's notice, that (2.17) the same appearance, as (2.16) has conditions. Really, let's replace  $\mu$  on  $-\mu$  in the condition (2.17). We will receive a condition

$$h(1, -\mu) = h(1, \mu) - A_0 - A_1\mu, \quad 0 < \mu < 1,$$

whence we receive in accuracy a condition (2.16). It means, that both conditions (2.16) and (2.17) it is possible to write down in the form of one

$$h(\pm 1, \mu) = h(\pm 1, -\mu) + A_0 + A_1\mu, \quad 0 < \mu < 1.$$

If in (2.16) we assume  $A_0 = A_1 = 0$ , then specular – accommodative boundary conditions pass into pure specular boundary conditions.

Coefficients  $A_0$  and  $A_1$  can be derived from the non-flowing condition and the definition of the normal electron momentum accommodation coefficient.

The problem statement is completed. Now the problem consists in finding of such solution of the system of equations (2.5) and (2.6), which satisfies the boundary conditions (2.7) and (2.16). Further, with the use of the solution of the problem, it is required to built the profiles of the distribution function of the electrons moving to the plasma surface, and profile of the electric field.

## 4 The relation between flows and boundary conditions

First of all let us find expression which relates the constants  $A_0$ ,  $A_1$  from the boundary condition (2.16). To carry this out we will use the condition of non-flowing (2.12) of the particle flow through the plasma boundary, which we will write as a sum of two flows

$$N_0 \equiv \int_0^1 \mu h(-1, \mu) d\mu + \int_{-1}^0 \mu h(-1, \mu) d\mu = 0.$$

After evident substitution of the variable in the second integral we obtain

$$N_0 \equiv \int_0^1 \mu \left[ h(-1, \mu) - h(-1, -\mu) \right] d\mu = 0.$$

Taking into account the relation (2.16), we obtain that  $A_0 = -\frac{2}{3}A_1$ .

With the help of this relation we can rewrite the condition (2.16) in the following form

$$h(-1, \mu) - h(-1, -\mu) = A_1(\mu - \frac{2}{3}), \quad 0 < \mu < 1, \quad (3.1)$$

or, that is equivalent,

$$h(1, \mu) - h(1, -\mu) = A_1(\mu - \frac{2}{3}), \quad 0 < \mu < 1, \quad (3.1')$$

We consider the momentum flow of the electrons which are moving to the boundary. According to (3.1) we have

$$P_i = P_r - \frac{1}{36}A_1. \quad (3.2)$$

It is easy to see further that

$$P_s = \frac{A_s}{3}. \quad (3.3)$$

With the help of the formulas (3.2) and (3.3) we will rewrite the definition of the accommodation coefficient (2.9) in the form

$$\alpha_p P_r - \alpha_p \frac{A_s}{3} + \frac{A_1}{36}(1 - \alpha_p) = 0. \quad (3.4)$$

Let us consider the condition (2.13). We rewrite it in the following form

$$\frac{A_s}{2} + \int_{-1}^0 \mu h(-1, \mu) d\mu = 0.$$

From this condition we obtain

$$A_s = -2 \int_{-1}^0 \mu h(-1, \mu) d\mu = 2 \int_0^1 \mu h(-1, -\mu) d\mu.$$

Using the condition (3.1), we then get

$$A_s = 2 \int_0^1 \mu h(-1, \mu) d\mu. \quad (3.5)$$

Now with the help of the second equality from (2.11) and (3.5) we rewrite the relation (3.4) in the integral form

$$\alpha_p \int_0^1 \left( \mu^2 - \frac{2}{3}\mu \right) h(-1, \mu) d\mu = -\frac{1}{36}(1 - \alpha_p)A_1. \quad (3.6)$$

Now the boundary problem consists of the equations (2.5) and (2.6) and boundary conditions (2.7), (3.1) and (3.6).

## 5 Separation of variables and characteristic system

Application of the general Fourier method of the separation of variables in several steps results in the following substitution [31]

$$h_\eta(x, \mu) = \exp\left(-\frac{w_0 x}{\eta}\right) \Phi(\eta, \mu) + \exp\left(\frac{z_0 x}{\eta}\right) \Phi(-\eta, \mu), \quad (4.1)$$

$$e_\eta(x) = \left[ \exp\left(-\frac{w_0 x}{\eta}\right) + \exp\left(\frac{w_0 x}{\eta}\right) \right] E(\eta), \quad (4.2)$$

where  $\eta$  is the spectrum parameter or the parameter of separation, which is complex in general.

We substitute the equalities (4.1) and (4.2) into the equations (2.5) and (2.6). We obtain the following characteristic system of equations

$$w_0(\eta - \mu) \Phi(\eta, \mu) = \eta \mu E(\eta) + \frac{\eta}{2} \int_{-1}^1 \Phi(\eta, \mu') d\mu', \quad (4.3)$$

$$w_0(\eta + \mu) \Phi(-\eta, \mu) = \eta \mu E(\eta) + \frac{\eta}{2} \int_{-1}^1 \Phi(-\eta, \mu') d\mu', \quad (4.4)$$

$$-\frac{w_0}{\eta} E(\eta) = u_p^2 \cdot \frac{1}{2} \int_{-1}^1 \Phi(\eta, \mu') d\mu', \quad (4.5)$$

$$\frac{w_0}{\eta} E(\eta) = u_p^2 \cdot \frac{1}{2} \int_{-1}^1 \Phi(-\eta, \mu') d\mu'. \quad (4.6)$$

From the equations (4.5) and (4.6) we obtain

$$\int_{-1}^1 \Phi(\eta, \mu) d\mu = - \int_{-1}^1 \Phi(-\eta, \mu) d\mu. \quad (4.7)$$

Let us introduce the designations

$$n(\eta) = \int_{-1}^1 \Phi(\eta, \mu) d\mu. \quad (4.8)$$

From the equation (4.5) we find, that

$$E(\eta) = -\frac{u_p^2}{2w_0} \eta n(\eta), \quad (4.9)$$

whence

$$n(\eta) = -2 \frac{w_0}{u_p^2} \cdot \frac{E(\eta)}{\eta}.$$

By means of equalities (4.7) – (4.9) we will copy the equations (4.3) and (4.4)

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{E(\eta)}{w_0}(\mu\eta - \eta_1^2), \quad (4.10)$$

$$(\eta + \mu)\Phi(-\eta, \mu) = \frac{E(\eta)}{w_0}(\mu\eta + \eta_1^2). \quad (4.11)$$

Here

$$\eta_1^2 = \frac{v_0 w_0}{u_p^2} = \frac{\nu(\nu - i\omega)}{3\omega_p^2} = \frac{\varepsilon^2 z_0}{3}, \quad \varepsilon = \frac{\nu}{\omega_p}, \quad z_0 = 1 - i\frac{\omega}{\nu} = 1 - i\frac{\Omega}{\varepsilon}, \quad \Omega = \frac{\omega}{\omega_p}.$$

Solution of the system (4.10) and (4.11) depends essentially on the condition whether the spectrum parameter  $\eta$  belongs to the interval  $-1 < \eta < 1$ . In connection with this the interval  $-1 < \eta < 1$  we will call as continuous spectrum of the characteristic system.

Let the parameter  $\eta \in (-1, 1)$ . Then from the equations (4.10) and (4.11) in the class of general functions we will find eigenfunction corresponding to the continuous spectrum

$$\Phi(\eta, \mu) = \frac{E(\eta)}{z_0} P \frac{\mu\eta - \eta_1^2}{\eta - \mu} + g_1(\eta)\delta(\eta - \mu), \quad (4.12)$$

$$\Phi(-\eta, \mu) = \frac{E(\eta)}{z_0} P \frac{\mu\eta + \eta_1^2}{\eta + \mu} + g_2(\eta)\delta(\eta - \mu). \quad (4.13)$$

In these equations (4.12) and (4.13)  $\delta(x)$  is the delta-function of Dirac, the symbol  $Px^{-1}$  means the principal value of the integral under integrating of the expression  $x^{-1}$ .

Substituting now (4.12) and (4.13) in the equations (4.5) and (4.6), we receive the equations from which we obtain

$$g_1(\eta) = -2 \frac{w_0}{u_p^2} \frac{\lambda(\eta)}{\eta} E(\eta), \quad g_2(\eta) = -g_1(\eta) = 2 \frac{w_0}{u_p^2} \frac{\lambda(\eta)}{\eta} E(\eta). \quad (4.14)$$

Here dispersion function is entered

$$\lambda(z) = 1 - \frac{z}{2c} \int_{-1}^1 \frac{\mu z - \eta_1^2}{\mu - z} d\mu, \quad (4.15)$$

where

$$c = \frac{w_0^2}{u_p^2} = \frac{\varepsilon^2 z_0^2}{3} = z_0 \eta_1^2.$$

Functions (4.12) and (4.13) are called eigen functions of the continuous spectrum, since the spectrum parameter  $\eta$  fills out the continuum  $(-1, +1)$  compactly. The eigen solutions of the given problem can be found from the equalities (4.1) and (4.2).

Substituting relations (4.14) in (4.12) and (4.13), we will present last expressions in the following form

$$\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} \left[ P \frac{\mu\eta - \eta_1^2}{\eta - \mu} - 2c \frac{\lambda(\eta)}{\eta} \delta(\eta - \mu) \right],$$

$$\Phi(-\eta, \mu) = \frac{E(\eta)}{w_0} \left[ P \frac{\mu\eta + \eta_1^2}{\eta + \mu} + 2c \frac{\lambda(\eta)}{\eta} \delta(\eta + \mu) \right],$$

or

$$\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} F(\eta, \mu), \quad \Phi(-\eta, \mu) = \frac{E(\eta)}{w_0} F(-\eta, \mu),$$

where

$$F(\eta, \mu) = P \frac{\mu\eta - \eta_1^2}{\eta - \mu} - 2c \frac{\lambda(\eta)}{\eta} \delta(\eta - \mu),$$

$$F(-\eta, \mu) = P \frac{\mu\eta + \eta_1^2}{\eta + \mu} + 2c \frac{\lambda(\eta)}{\eta} \delta(\eta + \mu).$$

It will be necessary for us the following relation of symmetry  $F(-\eta, -\mu) = -F(\eta, \mu)$ .

Let us notice, that eigen function  $F(\eta, \mu)$  satisfies to following condition of normalization

$$\int_{-1}^1 F(\pm\eta, \mu) d\mu = \mp 2c P \frac{1}{\eta}.$$

So, eigen function of a continuous spectrum is constructed and it is defined by equality

$$h_\eta(x, \mu) = \left[ \exp\left(-\frac{xw_0}{\eta}\right) F(\eta, \mu) + \exp\left(\frac{xw_0}{\eta}\right) F(-\eta, \mu) \right] \frac{E(\eta)}{w_0},$$

or, in explicit form,

$$h_\eta(x, \mu) = \left\{ \left[ \exp\left(-\frac{xw_0}{\eta}\right) \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \exp\left(\frac{xw_0}{\eta}\right) \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right] - \right. \\ \left. - 2c \frac{\lambda(\eta)}{\eta} \left[ \exp\left(-\frac{xw_0}{\eta}\right) \delta(\eta - \mu) - \exp\left(\frac{xw_0}{\eta}\right) \delta(\eta + \mu) \right] \right\} \frac{E(\eta)}{w_0}.$$

Let us replace exponents by hyperbolic functions

$$\exp\left(\frac{xw_0}{\eta}\right) = \text{ch} \frac{xw_0}{\eta} + \text{sh} \frac{xw_0}{\eta}, \quad \exp\left(-\frac{xw_0}{\eta}\right) = \text{ch} \frac{xw_0}{\eta} - \text{sh} \frac{xw_0}{\eta}$$

and also we will transform both square brackets from the previous expression. As a result we receive, that for the first square bracket it is had

$$\exp\left(-\frac{xw_0}{\eta}\right) \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \exp\left(\frac{xw_0}{\eta}\right) \frac{\mu\eta + \eta_1^2}{\eta + \mu} =$$

$$\begin{aligned}
&= \exp\left(-\frac{xw_0}{\eta}\right) \frac{(\mu\eta - \eta_1^2)(\eta + \mu)}{\eta^2 - \mu^2} + \exp\left(\frac{xw_0}{\eta}\right) \frac{(\mu\eta + \eta_1^2)(\eta - \mu)}{\eta^2 - \mu^2} = \\
&= 2 \left[ \operatorname{sh} \frac{xw_0}{\eta} P \frac{\eta(\mu^2 - \eta_1^2)}{\mu^2 - \eta} - \operatorname{ch} \frac{xw_0}{\eta} P \frac{\mu(\eta^2 - \eta_1^2)}{\mu^2 - \eta^2} \right].
\end{aligned}$$

For the second square bracket we have

$$\begin{aligned}
&\exp\left(-\frac{xw_0}{\eta}\right) \delta(\eta - \mu) - \exp\left(\frac{xw_0}{\eta}\right) \delta(\eta + \mu) = \\
&= - \left[ \delta(\eta - \mu) + \delta(\eta + \mu) \right] \operatorname{sh} \frac{xw_0}{\eta} + \left[ \delta(\eta - \mu) - \delta(\eta + \mu) \right] \operatorname{ch} \frac{xw_0}{\eta}.
\end{aligned}$$

As a result we receive, that

$$\begin{aligned}
h_\eta(x, \mu) = \frac{2E(\eta)}{w_0} &\left\{ \operatorname{ch} \frac{xw_0}{\eta} \left[ P \left( \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right) + \right. \right. \\
&+ 2c \frac{\lambda(\eta)}{\eta} (-\delta(\eta - \mu) + \delta(\eta + \mu)) \left. \right] + \operatorname{sh} \frac{xw_0}{\eta} \left[ P \left( -\frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1}{\eta + \mu} \right) + \right. \\
&\left. \left. + 2c \frac{\lambda(\eta)}{\eta} (\delta(\eta - \mu) + \delta(\eta + \mu)) \right] \right\}.
\end{aligned}$$

We will designate further

$$\varphi(\eta, \mu) = F(\eta, \mu) + F(-\eta, \mu) = P \left( \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right) = 2P \frac{\mu(\eta^2 - \eta_1^2)}{\eta^2 - \mu^2},$$

and

$$\psi(\eta, \mu) = -F(\eta, \mu) + F(-\eta, \mu) = P \left( -\frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1}{\eta + \mu} \right) = 2P \frac{\eta(\mu^2 - \eta_1^2)}{\eta^2 - \mu^2}.$$

Thus, eigen function of a continuous spectrum it is possible to present in the form of a linear combination of a hyperbolic sine and cosine

$$\begin{aligned}
h_\eta(x, \mu) = \frac{2E(\eta)}{w_0} &\left\{ \operatorname{ch} \frac{xw_0}{\eta} \left[ P \frac{\mu(\eta^2 - \eta_1^2)}{\eta^2 - \mu^2} - c \frac{\lambda(\eta)}{\eta} (\delta(\eta - \mu) - \delta(\eta + \mu)) \right] - \right. \\
&\left. - \operatorname{sh} \frac{xw_0}{\eta} \left[ P \frac{\eta(\mu^2 - \eta_1^2)}{\eta^2 - \mu^2} - c \frac{\lambda(\eta)}{\eta} (\delta(\eta - \mu) + \delta(\eta + \mu)) \right] \right\}.
\end{aligned}$$

Let us notice, that eigen functions of a continuous spectrum it is possible to present and in such form

$$h_\eta(x, \mu) = \frac{E(\eta)}{w_0} \left[ \operatorname{ch} \frac{xw_0}{\eta} \left( F(\eta, \mu) + F(-\eta, \mu) \right) + \operatorname{sh} \frac{xw_0}{\eta} \left( -F(\eta, \mu) + F(-\eta, \mu) \right) \right]. \quad (4.16)$$

The dispersion function  $\lambda(z)$  we express in the terms of the Case dispersion function [31]

$$\lambda(z) = 1 - \frac{1}{z_0} + \frac{1}{z_0} \left(1 - \frac{z^2}{\eta_1^2}\right) \lambda_C(z),$$

where

$$\lambda_C(z) = 1 + \frac{z}{2} \int_{-1}^1 \frac{d\tau}{\tau - z} = \frac{1}{2} \int_{-1}^1 \frac{\tau d\tau}{\tau - z}$$

is the Case dispersion function.

In the complex plane dispersion Case function is calculated through the logarithm

$$\lambda_C(z) = 1 + \frac{z}{2} \ln \frac{z-1}{z+1}, \quad z \in \mathbb{C} \setminus [-1, 1],$$

and on cut by formula

$$\lambda_0(\eta) = 1 + \frac{\eta}{2} \ln \frac{1-\eta}{1+\eta}, \quad \eta \in (-1, 1).$$

The boundary values of the dispersion function from above and below the cut (interval  $(-1, 1)$ ) we define in the following way

$$\lambda^\pm(\mu) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \lambda(\mu \pm i\varepsilon), \quad \mu \in (-1, 1).$$

The boundary values of the dispersion function from above and below the cut are calculated according to the Sokhotzky formulas

$$\lambda^\pm(\mu) = \lambda(\mu) \pm \frac{i\pi\mu}{2\eta_1^2 z_0} (\eta_1^2 - \mu^2), \quad -1 < \mu < 1,$$

from where

$$\lambda^+(\mu) - \lambda^-(\mu) = \frac{i\pi}{\eta_1^2 z_0} \mu (\eta_1^2 - \mu^2), \quad \frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} = \lambda(\mu), \quad -1 < \mu < 1,$$

where

$$\lambda(\mu) = 1 + \frac{\mu}{2\eta_1^2 z_0} \int_{-1}^1 \frac{\eta_1^2 - \eta^2}{\eta - \mu} d\eta,$$

and the integral in this equality is understood as singular in terms of the principal value by Cauchy. Besides that, the function  $\lambda(\mu)$  can be represented in the following form

$$\lambda(\mu) = 1 - \frac{1}{z_0} + \frac{1}{z_0} \left(1 - \frac{\mu^2}{\eta_1^2}\right) \lambda_0(\mu), \quad \lambda_0(\mu) = 1 + \frac{\mu}{2} \ln \frac{1-\mu}{1+\mu}.$$



## 6 Eigen functions of discrete spectrum and plasma waves

According to the definition, the discrete spectrum of the characteristic equation is a set of zeroes of the dispersion equation

$$\frac{\lambda(z)}{z} = 0. \quad (5.1)$$

We start to search zeroes of the equation (5.1). Let us take Laurent series of the dispersion function

$$\lambda(z) = \lambda_\infty + \frac{\lambda_2}{z^2} + \frac{\lambda_4}{z^4} + \dots, \quad |z| > 1. \quad (5.2)$$

Here

$$\lambda_\infty \equiv \lambda(\infty) = 1 - \frac{1}{z_0} + \frac{1}{3z_0\eta_1^2}, \quad \lambda_2 = -\frac{1}{z_0} \left( \frac{1}{3} - \frac{1}{5\eta_1^2} \right), \quad \lambda_4 = -\frac{1}{z_0} \left( \frac{1}{5} - \frac{1}{7\eta_1^2} \right).$$

We express these parameters through the parameters  $\gamma$  and  $\varepsilon$

$$\lambda_\infty \equiv \lambda(\infty) = \frac{2(\Omega - 1) + i\varepsilon + (\Omega - 1)(\Omega - 1 + i\varepsilon)}{(\Omega + i\varepsilon)^2},$$

$$\lambda_2 = -\frac{9 + 5i\varepsilon(\Omega + i\varepsilon)}{15(\Omega + i\varepsilon)^2}, \quad \lambda_4 = -\frac{15 + 7i\varepsilon(\Omega + i\varepsilon)}{35(\Omega + i\varepsilon)^2}.$$

It is easy seen that the dispersion function (4.9) in collisional plasma (i.e. when  $\varepsilon > 0$ ) in the infinity has the value which doesn't equal to zero:  $\lambda_\infty = \lambda(\infty) \neq 0$ .

Hence, the dispersion equation has infinity as a zero  $\eta_i = \infty$ , to which the discrete eigensolutions of the given system correspond

$$h_\infty(x, \mu) = \frac{\mu}{w_0}, \quad e_\infty(x) = 1.$$

This solution is naturally called as mode of Drude. It describes the volume conductivity of metal, considered by Drude (see, for example, [32]).

Let us consider the question of the plasma mode existence in details. We find finite complex zeroes of the dispersion function. We use the principle of argument. We take the contour (see Fig. 1)  $\Gamma_\varepsilon^+ = \Gamma_R \cup \gamma_\varepsilon$ ,  $\Gamma_R = \{z : |z| = R, \quad R = 1/\varepsilon, \quad \varepsilon > 0\}$ , which is passed in the positive direction the cut  $[-1, +1]$ , and which bounds the biconnected domain  $D_R$ .

Let us notice, that dispersion function in area  $D_R$  has no poles. Then owing to the principle of argument [33] zeroes number  $N$  in area  $D_R$  it is equal

$$N = \frac{1}{2\pi i} \oint_{\Gamma_\varepsilon} d \ln \lambda(z).$$

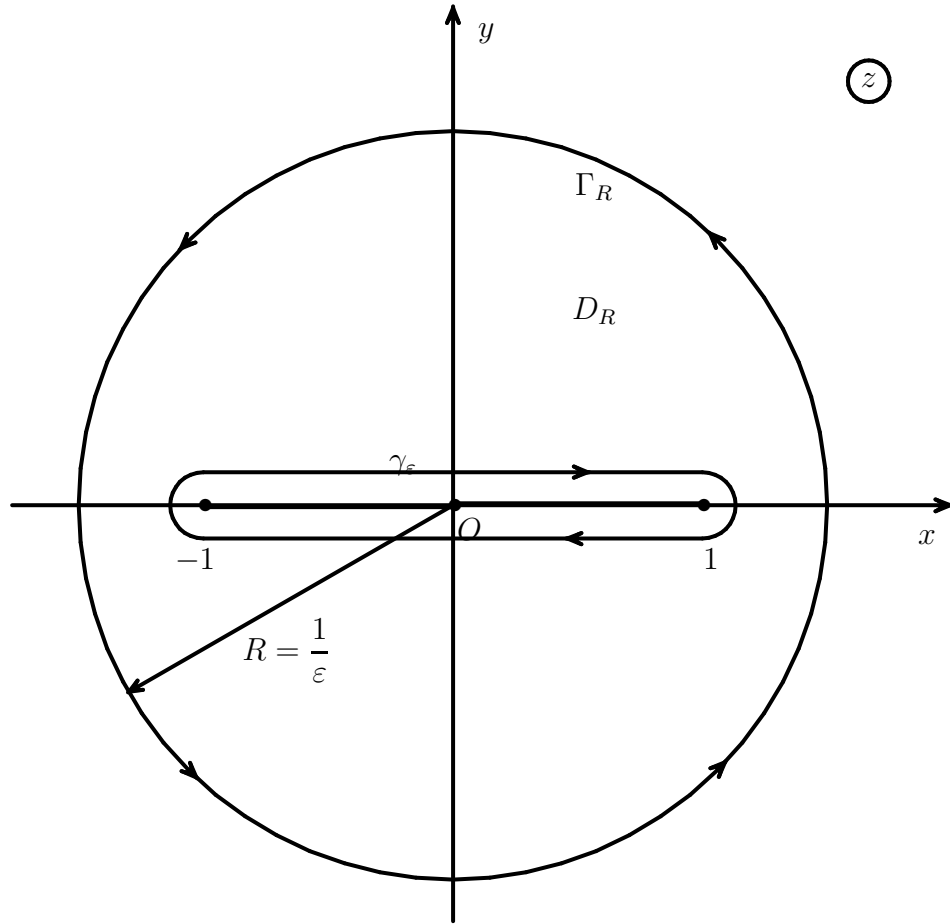


Fig 1. Contour  $\Gamma_\epsilon = \Gamma_R \cup \gamma_\epsilon$  for calculations of number of zero of dispersion function.

Considering the limit in this equality when  $\varepsilon \rightarrow 0$  and taking into account that the dispersion function is analytic in the neighbourhood of the infinity, we obtain that

$$N = \frac{1}{2\pi i} \int_{-1}^1 d \ln \lambda^+(\tau) - \frac{1}{2\pi i} \int_{-1}^1 d \ln \lambda^-(\tau) = \frac{1}{2\pi i} \int_{-1}^1 d \ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)}.$$

So, we have received, that

$$N = \frac{1}{2\pi i} \int_{-1}^1 d \ln \frac{\lambda^+(\tau)}{\lambda^-(\tau)}.$$

We divide this integral into two integrals by segments  $[-1, 0]$  and  $[0, 1]$ . In the first integral by the segment  $[-1, 0]$  we carry out replacement of variable  $\tau \rightarrow -\tau$ . Taking into account that  $\lambda^+(-\tau) = \lambda^-(\tau)$ , we obtain that

$$N = \frac{1}{2\pi i} \int_{-1}^1 d \ln \frac{\lambda^+(\tau)}{\lambda^-(\tau)} = \frac{1}{\pi i} \int_0^1 d \ln \frac{\lambda^+(\tau)}{\lambda^-(\tau)} = \frac{1}{\pi} \arg \frac{\lambda^+(\tau)}{\lambda^-(\tau)} \Big|_0^1. \quad (5.3)$$

Here under symbol  $\arg G(\tau) = \arg \frac{\lambda^+(\tau)}{\lambda^-(\tau)}$  we understand the regular branch of the argument, fixed in zero with the condition:  $\arg G(0) = 0$ .

We consider the curve  $\Lambda_G = \Lambda_G(\gamma, \varepsilon) : z = G(\tau)$ ,  $0 \leq \tau \leq +1$ , where

$$G(\tau) = \frac{\lambda^+(\tau)}{\lambda^-(\tau)}.$$

It is obvious that  $G(0) = 1$ ,  $\lim_{\tau \rightarrow +1} G(\tau) = 1$ . Consequently, according to (5.3), the number of values  $N$  equals to doubled number of turns of the curve  $\gamma$  around the point of origin, i.e.

$$N = 2\kappa(G), \quad (5.3')$$

where  $\kappa(G) = \text{Ind}_{[0, +1]} G(\tau)$  is the index of the function  $G(\tau)$ .

Thus, the number of zeroes of the dispersion function, which are situated in complex plane outside of the segment  $[-1, 1]$  of the real axis, equals to doubled index of the function  $G(\tau)$ , calculated on the "semi-segment"  $[0, +1]$ .

Let us single real and imaginary parts of the function  $G(\mu)$  out. At first, we represent the function  $G(\mu)$  in the form

$$G(\mu) = \frac{(z_0 - 1)\eta_1^2 + (\eta_1^2 - \mu^2)\lambda_0(\mu) + is(\mu)(\eta_1^2 - \mu^2)}{(z_0 - 1)\eta_1^2 + (\eta_1^2 - \mu^2)\lambda_0(\mu) - is(\mu)(\eta_1^2 - \mu^2)}.$$

where

$$s(\mu) = \frac{\pi}{2}\mu, \quad \lambda(\mu) = 1 - \frac{1}{z_0} + \frac{1}{z_0} \left(1 - \frac{\mu^2}{\eta_1^2}\right) \lambda_0(\mu),$$

and

$$\lambda_0(\mu) = 1 + \frac{\mu}{2} \ln \frac{1-\mu}{1+\mu}$$

is the dispersion function of Case, calculated on the cut (i.e., in the interval  $(-1, 1)$ ).

Taking into account that

$$z_0 - 1 = -i\frac{\omega}{\nu} = -i\frac{\Omega}{\varepsilon}, \quad \eta_1^2 = \frac{\varepsilon z_0}{3} = \frac{\varepsilon^2}{3} - i\frac{\varepsilon\Omega}{3}, \quad (z_0 - 1)\eta_1^2 = -\frac{\Omega^2}{3} - i\frac{\varepsilon\Omega}{3},$$

we obtain

$$G(\mu) = \frac{P^-(\mu) + iQ^-(\mu)}{P^+(\mu) + iQ^+(\mu)},$$

where

$$P^\pm(\mu) = \Omega^2 - \lambda_0(\mu)(\varepsilon^2 - 3\mu^2) \pm \varepsilon\Omega s(\mu),$$

$$Q^\pm(\mu) = \varepsilon\Omega(1 + \lambda_0(\mu)) \pm s(\mu)(\varepsilon^2 - 3\mu^2).$$

Now we can easily separate real and imaginary parts of the function  $G(\mu)$

$$G(\mu) = \frac{g_1(\mu)}{g(\mu)} + i\frac{g_2(\mu)}{g(\mu)}.$$

Here

$$\begin{aligned} g(\mu) &= [P^+(\mu)]^2 + [Q^+(\mu)]^2 = \\ &+ [\Omega^2 + \lambda_0(3\mu^2 - \varepsilon^2) - \varepsilon\Omega s]^2 + [\varepsilon\Omega(1 + \lambda_0) - s(3\mu^2 - \varepsilon^2)]^2, \\ g_1(\mu) &= P^+(\mu)P^-(\mu) + Q^+(\mu)Q^-(\mu) = \\ &= [\Omega^2 + \lambda_0(3\mu^2 - \varepsilon^2)]^2 - \varepsilon^2\Omega^2[s^2 - (1 + \lambda_0)^2] - (3\mu^2 - \varepsilon^2)^2 s^2, \\ g_2(\mu) &= P^+(\mu)Q^-(\mu) - P^-(\mu)Q^+(\mu) = \\ &+ 2s[\Omega^2(3\mu^2 - \varepsilon^2) + \lambda_0(3\mu^2 - \varepsilon^2)^2 + \varepsilon^2\Omega^2(1 + \lambda_0)], \end{aligned}$$

We consider (see Fig. 2) the curve  $L$ , which is defined in implicit form by the following parametric equations

$$L = \{(\Omega, \varepsilon) : \quad g_1(\mu; \Omega, \varepsilon) = 0, \quad g_2(\mu; \Omega, \varepsilon) = 0, \quad 0 \leq \mu \leq 1\},$$

and which lays in the plane of the parameters of the problem  $(\gamma, \varepsilon)$ , and when passing through this curve the index of the function  $G(\mu)$  at the positive "semi-segment"  $[0, 1]$  changes stepwise.

From the equation  $g_2 = 0$  we find

$$\Omega^2 = -\frac{\lambda_0(\mu)(3\mu^2 - \varepsilon^2)}{3\mu^2 + \varepsilon^2\lambda_0(\mu)}. \quad (5.4)$$

Now from the equation  $g_1 = 0$  with the help of (5.4) we find that

$$\varepsilon = \sqrt{L_2(\mu)}, \quad (5.5)$$

where

$$L_2(\mu) = -\frac{3\mu^2 s^2(\mu)}{\lambda_0(\mu)[s^2(\mu) + (1 + \lambda_0(\mu))^2]}.$$

Substituting (5.5) into (5.4), we obtain

$$\Omega = +\sqrt{L_1(\mu)}, \quad (5.6)$$

where

$$L_1(\mu) = -\frac{3\mu^2[s^2(\mu) + \lambda_0(\mu)(1 + \lambda_0(\mu))]^2}{\lambda_0(\mu)[s^2(\mu) + (1 + \lambda_0(\mu))^2]}.$$

Functions (5.5) and (5.6) determine the curve  $L$  which is the border of the domain  $D^+$  (we designate the external area to the domain as  $D^-$ ) in explicit parametrical form (see Fig. 2). As in the work [34] we can prove that if  $(\gamma, \varepsilon) \in D^+$ , then  $\varkappa(G) = \text{Ind}_{[0, +1]} G(\mu) = 1$  (the curve  $L$  encircles the point of origin once), and if  $(\gamma, \varepsilon) \in D^-$ , then  $\varkappa(G) = \text{Ind}_{[0, +1]} G(\mu) = 0$  (the curve  $L$  doesn't encircle the point of origin).

We note, that in the work [34] the method of analysis of boundary regime when  $(\gamma, \varepsilon) \in L$  was developed.

From the expression (3.2) one can see that the number of zeroes of the dispersion function equals to two if  $(\gamma, \varepsilon) \in D^+$ , and equals to zero if  $(\gamma, \varepsilon) \in D^-$ .

Since the dispersion function is even its zeroes differ from each other by sign. We designate these zeroes as following  $\pm\eta_0$ , by  $\eta_0$  we take the zero which satisfies the condition  $\text{Re } \eta_0 > 0$ .

The following solution corresponds to the zero  $\pm\eta_0$

$$h_{\eta_0}(x, \mu) = \text{ch } \frac{xw_0}{\eta_0} [F(\eta_0, \mu) + F(-\eta_0, \mu)] + \text{sh } \frac{xw_0}{\eta_0} [-F(\eta_0, \mu) + F(-\eta_0, \mu)], \quad (5.7)$$

$$e_{\eta_0}(x) = 2 \text{ch } \frac{w_0 x}{\eta_0}. \quad (5.8)$$

Here

$$F(\eta_0, \mu) = \frac{\eta_0 \mu - \eta_1^2}{\eta_0 - \mu}, \quad F(-\eta_0, \mu) = \frac{\eta_0 \mu + \eta_1^2}{\eta_0 + \mu}.$$

It is easy to see, that function  $h_{\eta_0}(x, \mu)$  is even on  $\eta_0$

$$h_{\eta_0}(x, \mu) = h_{-\eta_0}(x, \mu).$$

Function  $h_{\eta_0}(x, \mu)$  we will present in the explicit form

$$h_{\eta_0}(x, \mu) = \text{ch } \frac{xw_0}{\eta_0} \left[ \frac{\eta_0 \mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0 \mu + \eta_1^2}{\eta_0 + \mu} \right] + \text{sh } \frac{xw_0}{\eta_0} \left[ -\frac{\eta_0 \mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0 \mu + \eta_1^2}{\eta_0 + \mu} \right], \quad (5.9)$$

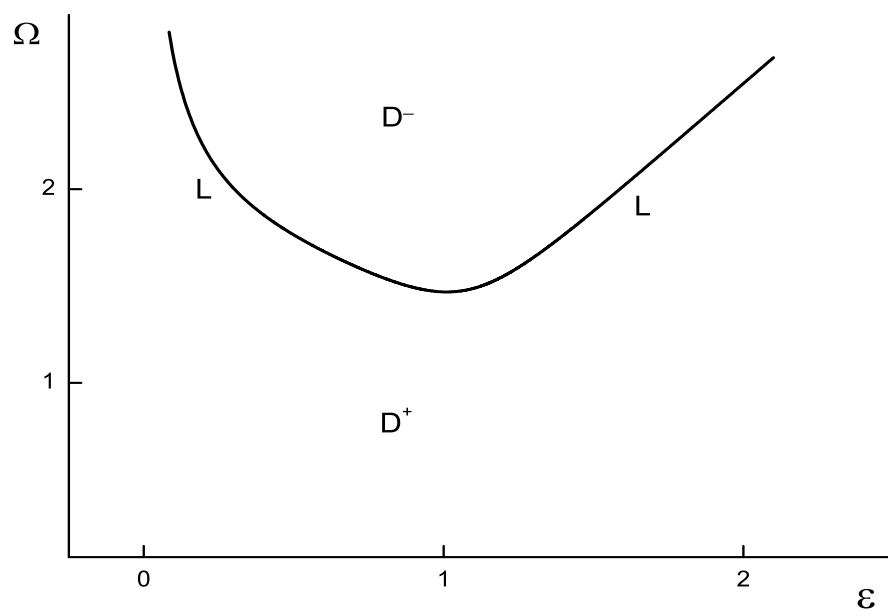


Fig. 2.

or

$$h_{\eta_0}(x, \mu) = \text{ch} \frac{xw_0}{\eta_0} \varphi(\eta_0, \mu) + \text{sh} \frac{xw_0}{\eta_0} \psi(\eta_0, \mu). \quad (5.9')$$

Here

$$\begin{aligned} \varphi(\eta_0, \mu) &= F(\eta_0, \mu) + F(-\eta_0, \mu) = \frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} = \frac{2\mu(\eta_0^2 - \eta_1^2)}{\eta_0^2 - \mu^2}, \\ \psi(\eta_0, \mu) &= -F(\eta_0, \mu) + F(-\eta_0, \mu) = -\frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} = \frac{2\eta_0(\eta_1^2 - \mu^2)}{\eta_0^2 - \mu^2}. \end{aligned}$$

This solution is naturally called as mode of Debay (this is plasma mode). In the case of low frequencies it describes well-known screening of Debay [3]. The external field penetrates into plasma on the depth of  $r_D$ ,  $r_D$  is the raduis of Debay. When the external field frequencies are close to Langmuir frequencies, the mode of Debay describes plasma oscillations (see, for instance, [3, 32]).

**Note 5.1.** If to enter expression  $c/z$  "inside" of expression of dispersion function we will receive expression for dispersion function  $h(z)$  from our article [21]

$$h(z) = \frac{c}{z} \lambda(z) = \frac{c}{z} - \frac{1}{2} \int_{-1}^1 \frac{\mu z - \eta_1^2}{\mu - z} d\mu = \frac{c}{z} - z - (z^2 - \eta_1^2) \frac{1}{2} \ln \frac{z-1}{z+1}.$$

## 7 Expansions by eigen functions

We will seek for the solution of the system of equations (2.5) and (2.6) with boundary conditions (3.1), (3.6) and (2.7) in the form of linear combination of discrete eigen solutions of the characteristic system and integral taken over continuous spectrum of the system. Let us prove that the following theorem is true.

**Theorem 6.1.** *System of equations (2.5) and (2.6) with boundary conditions (3.1), (3.6) and (2.7) has a unique solution, which can be presented as an expansion by eigen functions of the characteristic system*

$$\begin{aligned} h(x, \mu) &= \frac{E_\infty}{w_0} \mu + \frac{E_0}{w_0} \left[ \frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} \exp\left(-\frac{w_0x}{\eta_0}\right) + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} \exp\left(\frac{w_0x}{\eta_0}\right) \right] + \\ &+ \int_{-1}^1 \left[ \exp\left(-\frac{w_0x}{\eta}\right) F(\eta, \mu) + \exp\left(\frac{w_0x}{\eta}\right) F(-\eta, \mu) \right] \frac{E(\eta)}{w_0} d\eta, \quad (6.1) \\ e(x) &= E_\infty + E_0 \left[ \exp\left(-\frac{w_0x}{\eta}\right) + \exp\left(\frac{w_0x}{\eta_0}\right) \right] + \end{aligned}$$

$$+ \int_{-1}^1 \left[ \exp\left(-\frac{w_0 x}{\eta}\right) + \exp\left(\frac{w_0 x}{\eta}\right) \right] E(\eta) d\eta. \quad (6.2)$$

Here  $E_0$  and  $E_\infty$  is unknown coefficients corresponding to the discrete spectrum ( $E_0$  is the amplitude of Debay,  $E_1$  is the amplitude of Drude),  $E(\eta)$  is unknown function, which is called as coefficient of continuous spectrum.

When  $(\Omega, \varepsilon) \in D^-$  in expansions (6.1) and (6.2) we should take  $E_0 = 0$ .

Further we will consider the following case  $(\Omega, \varepsilon) \in D^+$ .

Our purpose is to find the coefficient of the continuous spectrum, coefficients of the discrete spectrum and to built expressions for electron distribution function at the plasma surface and electric field.

**Proof.** Let us notice, that the formula (6.1) can be transformed and to such form

$$\begin{aligned} h(x, \mu) = & \frac{E_\infty}{w_0} \mu + \frac{2E_0}{w_0} \left[ \frac{\eta_0(\mu^2 - \eta_1^2)}{\mu^2 - \eta_0^2} \operatorname{sh} \frac{w_0 x}{\eta_0} - \frac{\mu(\eta_0^2 - \eta_1^2)}{\mu^2 - \eta_0^2} \operatorname{ch} \frac{w_0 x}{\eta_0} \right] + \\ & + \frac{1}{w_0} \int_{-1}^1 \left[ \operatorname{ch} \frac{w_0 x}{\eta} F(\eta, \mu) + \operatorname{sh} \frac{w_0 x}{\eta} F(-\eta, \mu) \right] E(\eta) d\eta. \end{aligned} \quad (6.1')$$

Let us consider expansion (6.1), we will replace in it  $\mu$  on  $-\mu$ . Then we will substitute the difference  $h(1, \mu) - h(1, -\mu)$  in a boundary condition (3.1). After variety of transformations let us have

$$\begin{aligned} & E_\infty \mu + E_0 \left[ F(\eta_0, \mu) + F(-\eta_0, \mu) \right] \operatorname{ch} \frac{w_0}{\eta_0} + \\ & + \int_{-1}^1 F(\eta, \mu) E(\eta) \operatorname{ch} \frac{w_0}{\eta} d\eta = \frac{z_0 A_1}{2} \left( \mu - \frac{2}{3} \right), \quad 0 < \mu < 1. \end{aligned} \quad (6.3)$$

Substituting expansion (6.2) in (2.7), we will have

$$E_\infty + 2E_0 \operatorname{ch} \frac{w_0}{\eta_0} + 2 \int_{-1}^1 E(\eta) \operatorname{ch} \frac{w_0}{\eta} d\eta = 1. \quad (6.4)$$

Let us pass from Fredholm integral equation (6.3) to singular integral equation with Cauchy kernel, having substituted in (6.3) obvious representation  $F(\eta, \mu)$

$$\begin{aligned} & E_\infty \mu + E_0 \varphi(\eta_0, \mu) \operatorname{ch} \frac{w_0}{\eta_0} + \int_{-1}^1 \frac{\mu \eta - \eta_1^2}{\eta - \mu} E(\eta) \operatorname{ch} \frac{w_0}{\eta} d\eta - 2c \frac{\lambda(\mu)}{\mu} \operatorname{ch} \frac{w_0}{\mu} E(\mu) = \\ & = \frac{w_0 A_1}{2} \left( \mu - \frac{2}{3} \right), \quad 0 < \mu < 1, \end{aligned} \quad (6.5)$$



where

$$\varphi(\eta_0, \mu) = F(\eta_0, \mu) + F(-\eta_0, \mu) = \frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} = 2\mu \frac{\eta_0^2 - \eta_1^2}{\eta_0^2 - \mu^2}.$$

It is easy to check up, that function

$$M(z) = \int_{-1}^1 \frac{z\eta - \eta_1^2}{\eta - z} \operatorname{ch} \frac{w_0}{\eta} E(\eta) d\eta \quad (6.6)$$

is odd. Besides, all members of the equation (6.5) are odd on  $\mu$ , except one member from its right part  $-\frac{1}{3}w_0A_1$ .

Hence, the equation (6.5) can be extended in the interval  $-1 < \mu < 1$  in the next symmetric form

$$\begin{aligned} E_\infty\mu + E_0\varphi(\eta_0, \mu) \operatorname{ch} \frac{w_0}{\mu} + \int_{-1}^1 \operatorname{ch} \frac{w_0}{\eta} \frac{\mu\eta - \eta_1^2}{\eta - \mu} E(\eta) d\eta - \\ - 2c \frac{\lambda(\mu)}{\mu} \operatorname{ch} \frac{w_0}{\mu} E(\mu) - \frac{w_0A_1}{2}\mu = -\frac{1}{3}w_0A_1 \operatorname{sign} \mu, \quad -1 < \mu < 1. \end{aligned} \quad (6.7)$$

Let us reduce the equation (6.7) to Riemann — Hilbert boundary value problem. For this purpose we will take advantage Sohkotsky formulas for the auxiliary functions  $M(z)$  and dispersion function  $\lambda(z)$

$$M^+(\mu) - M^-(\mu) = 2\pi i \operatorname{ch} \frac{w_0}{\mu} (\mu^2 - \eta_1^2) E(\mu), \quad -1 < \mu < 1, \quad (6.8)$$

$$\frac{M^+(\mu) + M^-(\mu)}{2} = M(\mu), \quad -1 < \mu < 1,$$

where

$$M(\mu) = \int_{-1}^1 \frac{\mu\eta - \eta_1^2}{\eta - \mu} \operatorname{ch} \frac{w_0}{\eta} E(\eta) d\eta,$$

besides, last integral is understood as singular in the sense of principal value of Cauchy, and

$$\lambda^+(\mu) - \lambda^-(\mu) = -\frac{i\pi}{c} \mu(\mu^2 - \eta_1^2), \quad \frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} = \lambda(\mu), \quad -1 < \mu < 1,$$

where

$$\lambda(\mu) = 1 - \frac{\mu}{2c} \int_{-1}^1 \frac{\mu'\mu - \eta_1^2}{\mu' - \mu} d\mu',$$

besides, last integral is understood as singular in the sense of principal value of Cauchy also.

As result of use of last formulas we will come to the boundary value problem

$$E_\infty \mu + E_0 \varphi(\eta_0, \mu) \operatorname{ch} \frac{w_0}{\eta_0} + \frac{1}{2} [M^+(\mu) + M^-(\mu)] + \frac{1}{2} \frac{\lambda^+(\mu) + \lambda^-(\mu)}{\lambda^+(\mu) - \lambda^-(\mu)} [M^+(\mu) - M^-(\mu)] -$$

$$-\frac{w_0 A_1}{2} \mu = -\frac{1}{3} w_0 A_1 \operatorname{sign} \mu, \quad -1 < \mu < 1.$$

We transform this equation to the form

$$\frac{1}{2} (M^+ + M^-)(\lambda^+ - \lambda^-) + \frac{1}{2} (M^+ - M^-)(\lambda^+ + \lambda^-) +$$

$$+(\lambda^+ - \lambda^-) \left( E_\infty \mu + E_0 \varphi(\eta_0, \mu) \operatorname{ch} \frac{w_0}{\eta_0} - \frac{w_0 A_1}{2} \mu \right) =$$

$$= \frac{1}{3} w_0 A_1 \operatorname{sign} \mu \frac{i\pi}{c} \mu (\mu^2 - \eta_1^2), \quad -1 < \mu < 1.$$

From here we receive the following boundary condition of boundary value Riemann — Hilbert

$$\lambda^+(\mu) \left[ M^+(\mu) + E_\infty \mu + E_0 \varphi(\eta_0, \mu) \operatorname{ch} \frac{w_0}{\eta_0} - \frac{w_0 A_1}{2} \mu \right] -$$

$$= \lambda^-(\mu) \left[ M^-(\mu) + E_\infty \mu + E_0 \varphi(\eta_0, \mu) \operatorname{ch} \frac{w_0}{\eta_0} - \frac{w_0 A_1}{2} \mu \right] =$$

$$= \frac{\pi i}{3c} w_0 A_1 \mu \operatorname{sign} \mu (\mu^2 - \eta_1^2), \quad -1 < \mu < 1. \quad (6.9)$$

We rewrite this problem in the form

$$\Phi^+(\mu) - \Phi^-(\mu) = \frac{i\pi}{3c} w_0 A_1 \mu (\mu^2 - \eta_1^2) \operatorname{sign} \mu, \quad -1 < \mu < 1. \quad (6.10)$$

In the problem (6.10)  $\Phi^\pm(\mu)$  are boundary values on interval  $-1 < \mu < 1$  of function

$$\Phi(z) = \lambda^+(z) \left[ M^+(z) + E_\infty z + E_0 \varphi(\eta_0, z) \operatorname{ch} \frac{w_0}{\eta_0} - \frac{w_0 A_1}{2} z \right],$$

which is analytic in complex plane with cut  $\mathbb{C} \setminus [-1, 1]$ .

The problem (6.10) is a problem of special case about jump. The jump problem is the problem of finding of analytical function by its jump on the contour  $L$ :

$$\Phi^+(\mu) - \Phi^-(\mu) = \varphi(\mu), \quad \mu \in L.$$

The solution of such problems in a class of functions decreasing at infinitely remote point it is given by integral of Cauchy type

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) d\tau}{\tau - z}.$$

However, in the problem (6.10), when

$$\varphi(\mu) = \frac{2i\pi}{6c} w_0 A_1 \mu (\mu^2 - \eta_1^2) \operatorname{sign} \mu, \quad -1 < \mu < 1,$$

unknown function  $\Phi(z)$  has at infinitely remote point the following asymptotic

$$\Phi(z) = O(z), \quad z \rightarrow \infty.$$

Therefore it is necessary to search the solution of the problem (6.11) in a class of the growing as  $z$  in the vicinity of infinitely remote point.

According to [33] the general solution of problem (6.10) it is given by the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) d\tau}{\tau - z} + C_1 z.$$

In explicit form the general solution of problem (6.10) write down the following form

$$\begin{aligned} \lambda(z) \left[ M(z) + E_0 \varphi(\eta_0, z) \operatorname{ch} \frac{w_0}{\eta_0} + E_\infty w_0 - \frac{w_0 A_1}{2} z \right] = \\ + \frac{1}{3} (w_0 A_1) \frac{1}{2c} \int_{-1}^1 \frac{\mu (\mu^2 - \eta_1^2) \operatorname{sign} \mu}{\mu - z} d\mu + C_1 z, \end{aligned}$$

where  $C_1$  is the arbitrary constant.

Let us enter auxiliary function

$$T(z) = \frac{1}{2c} \int_{-1}^1 \frac{\mu (\mu^2 - \eta_1^2) \operatorname{sign} \mu}{\mu - z} d\mu.$$

Then the general solution receives the following form

$$\lambda(z) \left[ M(z) + E_0 \varphi(z) + E_\infty z - \frac{w_0 A_1}{2} z \right] = \frac{1}{3} w_0 A_1 T(z) + C_1 z.$$

From this general solution we can find the function  $M(z)$

$$M(z) = -E_\infty z - E_0 \varphi(\eta_0, z) \operatorname{ch} \frac{w_0}{\eta_0} + \frac{w_0 A_1}{2} z + \frac{1}{3} w_0 A_1 \frac{T(z)}{\lambda(z)} + \frac{C_1 z}{\lambda(z)}. \quad (6.11)$$

Let us remove a pole at the solution (6.11) at infinitely remote point. Let us notice, that the function

$$\varphi(\eta_0, z) = \frac{2z(\eta_1^2 - \eta_0^2)}{z^2 - \eta_0^2}.$$

at  $z \rightarrow \infty$  has the following asymptotic

$$\varphi(\eta_0, z) = O\left(\frac{1}{z}\right), \quad z \rightarrow \infty.$$

Considering, that function  $T(z)$  has exactly the same asymptotic at

$$C_1 = \left(E_\infty - \frac{z_0 A_1}{2}\right) \lambda_\infty. \quad (6.12)$$

## 8 Coefficients of discrete and continuous spectra

Now we will remove poles at the solution (6.11) at the points  $\pm\eta_0$ . Let us allocate in the right part of the solution (6.11) members containing the polar singularity at the point  $z = \eta_0$ . In the point vicinity  $z = \eta_0$  taking into account equality  $\lambda(\eta_0) = 0$  it is carried out the following expansion

$$M(z) = -\left(E_\infty - \frac{w_0 A_1}{2}\right)z - E_0 \frac{\eta_1^2 + \eta_0 z}{z + \eta_0} \operatorname{ch} \frac{w_0}{\eta_0} + \\ + \frac{1}{z - \eta_0} \left[ -E_0(\eta_1^2 - \eta_0^2) \operatorname{ch} \frac{w_0}{\eta_0} + \frac{(1/3)w_0 A_1 T(z) + \eta_0(E_\infty - w_0 A_1/2)\lambda_\infty}{\lambda'(\eta_0) + (1/2!)\lambda''(\eta_0)(z - \eta_0) + \dots} \right].$$

From here it is visible, that for pole elimination at the point  $z = \eta_0$  it is necessary to equate to zero expression in a square bracket, calculated at  $z = \eta_0$ . Then we receive, that

$$w_0 A_1 = \frac{E_0 \lambda'(\eta_0)(\eta_1^2 - \eta_0^2) \operatorname{ch}(w_0/\eta_0) - \eta_0 E_\infty \lambda_\infty}{T(\eta_0)/3 - \eta_0 \lambda_\infty/2}, \quad (7.1)$$

therefore

$$2E_0 \operatorname{ch} \frac{w_0}{\eta_0} = \frac{w_0 A_1 [2T(\eta_0)/3 - \lambda_\infty \eta_0] + 2\eta_0 \lambda_\infty E_\infty}{\lambda'(\eta_0)(\eta_1^2 - \eta_0^2)}. \quad (7.2)$$

The continuous spectra coefficient  $E(\eta)$  we find from formulas (6.11) and (6.12)

$$E(\eta) = \frac{1}{2\pi i} \cdot \frac{M^+(\eta) - M^-(\eta)}{\operatorname{ch} \frac{w_0}{\eta} (\eta^2 - \eta_1^2)}. \quad (7.3)$$

Difference  $M^+(\eta) - M^-(\eta)$  from the formula (7.3) we will find with the help formulas of the general solution (6.12). As result we receive the following expression

$$E(\eta) = \frac{1}{2\pi i \operatorname{ch} \frac{w_0}{\eta} (\eta^2 - \eta_1^2)} \left\{ \frac{1}{3} w_0 A_1 \left[ \frac{T^+(\eta)}{\lambda^+(\eta)} - \frac{T^-(\eta)}{\lambda^-(\eta)} \right] + \right. \\ \left. + \left( E_\infty - \frac{w_0 A_1}{2} \right) \lambda_\infty \eta \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] \right\}. \quad (7.4)$$

Let us substitute equalities (7.2) and (7.4) in the equation on the field (6.4). We receive the following equation

$$\lambda_\infty E_\infty \left( \frac{1}{\lambda_\infty} - \frac{2\eta_0}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} + J_1 \right) + w_0 A_1 \left( -\frac{2T(\eta_0)/3 - \lambda_\infty \eta_0}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} - \frac{\lambda_\infty}{2} J_1 + \frac{1}{3} J_2 \right) = 1, \quad (7.5)$$

where

$$J_1 = \frac{1}{2\pi i} \int_{-1}^1 \left( \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right) \frac{\eta d\eta}{\eta^2 - \eta_1^2},$$

$$J_2 = \frac{1}{2\pi i} \int_{-1}^1 \left( \frac{T^+(\eta)}{\lambda^+(\eta)} - \frac{T^-(\eta)}{\lambda^-(\eta)} \right) \frac{d\eta}{\eta^2 - \eta_1^2}.$$

Integrals  $J_1$  and  $J_2$  from the equation (7.5) can be calculated analytically by means of the theory of residues and contour integration. For the first integral we have

$$\begin{aligned} J_1 &= \frac{1}{2\pi i} \int_{-1}^1 \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] \frac{\eta d\eta}{\eta^2 - \eta_1^2} = \\ &= \left[ \text{Res}_{\infty} + \text{Res}_{\eta_1} + \text{Res}_{-\eta_1} + \text{Res}_{\eta_0} + \text{Res}_{-\eta_0} \right] \frac{z}{(z^2 - \eta_1^2)\lambda(z)}. \end{aligned}$$

Let us notice, that

$$\begin{aligned} \text{Res}_{\infty} \frac{z}{(z^2 - \eta_1^2)\lambda(z)} &= -\frac{1}{\lambda_{\infty}}, \quad \text{Res}_{\pm\eta_1} \frac{z}{(z^2 - \eta_1^2)\lambda(z)} = \frac{1}{2\lambda(\eta_1)}, \\ \text{Res}_{\pm\eta_0} \frac{z}{(z^2 - \eta_1^2)\lambda(z)} &= \frac{\eta_0}{(\eta_0^2 - \eta_1^2)\lambda'(\eta_0)}. \end{aligned}$$

Hence, the integral is equal

$$J_1 = -\frac{1}{\lambda_{\infty}} + \frac{1}{\lambda_1} + \frac{2\eta_0}{(\eta_0^2 - \eta_1^2)\lambda'(\eta_0)},$$

where

$$\lambda_1 = \lambda(\eta_1) = 1 - \frac{1}{z_0}.$$

In the same way we calculate the second integral

$$\begin{aligned} J_2 &= \frac{1}{2\pi i} \int_{-1}^1 \frac{T_2(\eta)}{\eta^2 - \eta_1^2} \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] d\eta = \\ &= \left[ \text{Res}_{\infty} + \text{Res}_{\eta_1} + \text{Res}_{-\eta_1} + \text{Res}_{\eta_0} + \text{Res}_{-\eta_0} \right] \frac{T(z)}{(z^2 - \eta_1^2)\lambda(z)}. \end{aligned}$$

Let us notice, that

$$\begin{aligned} \text{Res}_{\infty} \frac{T(z)}{(z^2 - \eta_1^2)\lambda(z)} &= 0, \quad \text{Res}_{\pm\eta_1} \frac{T(z)}{(z^2 - \eta_1^2)\lambda(z)} = \pm \frac{T(\pm\eta_1)}{2\eta_1\lambda_1}, \\ \text{Res}_{\pm\eta_0} \frac{T(z)}{(z^2 - \eta_1^2)\lambda(z)} &= \pm \frac{T(\pm\eta_0)}{(\eta_0^2 - \eta_1^2)\lambda'(\eta_0)}. \end{aligned}$$

Hence, the integral is equal

$$J_2 = \frac{T(\eta_1) - T(-\eta_1)}{2\eta_1\lambda_1} + \frac{T(\eta_0) - T(-\eta_0)}{(\eta_0^2 - \eta_1^2)\lambda'(\eta_0)}.$$

Let us notice, that  $T(\eta_1) - T(-\eta_1) = \frac{\eta_1}{c}$ . Thus, it is definitively received

$$J_2 = \frac{1}{2c\lambda_1} + \frac{2T(\eta_0)}{(\eta_0^2 - \eta_1^2)\lambda'(\eta_0)}.$$

By means of these integrals the condition on the field (7.5) transforms to the form

$$\frac{\lambda_\infty E_\infty}{\lambda_1} + \frac{w_0 A_1}{2\lambda_1} \left( \frac{1}{3c} + \lambda_1 - \lambda_\infty \right) = 1, \quad \text{or} \quad \frac{\lambda_\infty E_\infty}{\lambda_1} = 1,$$

because  $\lambda_\infty = \lambda_1 + 1/3c$ .

Hence, from last equation we find the Drude amplitude

$$E_\infty = \frac{\lambda_1}{\lambda_\infty}. \quad (7.6)$$

Now the condition (7.2) will be transformed to the form

$$E_0 = -\frac{\lambda_1 \eta_0 + w_0 A_1 (\frac{1}{3}T(\eta_0) - \frac{1}{2}\lambda_\infty \eta_0)}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2) \text{ch}(w_0/\eta_0)}. \quad (7.7)$$

## 9 Integral condition on distribution function

Let us notice, that at transition through the positive part of the cut  $(0, 1)$  functions  $T(z)$  and  $\lambda(z)$  make the jump, differing only by signs. Really, the formula for  $T(z)$  we will present in the kind

$$T(z) = \frac{1}{2c} \int_0^1 \mu(\mu^2 - \eta_1^2) \left[ \frac{1}{\mu - z} - \frac{1}{\mu + z} \right] d\mu = \frac{z}{c} \int_0^1 \frac{\mu(\mu^2 - \eta_1^2)}{\mu^2 - z^2} d\mu,$$

or

$$T(z) = \frac{z}{2c} \int_0^1 (\mu^2 - \eta_1^2) \left[ \frac{1}{\mu - z} + \frac{1}{\mu + z} \right] d\mu = \frac{z}{c} \int_0^1 \frac{\mu(\mu^2 - \eta_1^2)}{\mu^2 - z^2} d\mu.$$

This integral is easy for calculating in an explicit form. In a complex plane with a cut function  $T(z)$  is calculated by the formula

$$T(z) = \frac{z}{2c} \left[ 1 + (z^2 - \eta_1^2) \ln \left( 1 - \frac{1}{z^2} \right) \right], \quad z \in \mathbb{C},$$

and on the cut this integral is calculated by formula

$$T(\eta) = \frac{\eta}{2c} \left[ 1 + (\eta^2 - \eta_1^2) \ln \left( \frac{1}{\eta^2} - 1 \right) \right], \quad -1 < \eta < +1.$$

Now from the Sokhotsky formula for a difference of boundary values we receive, that at  $0 < \eta < 1$

$$\lambda^+(\eta) = \lambda(\eta) \pm \frac{i\pi}{2c} \eta(\eta_1^2 - \eta^2), \quad T^+(\eta) = T(\eta) \pm \frac{i\pi}{2c} \eta(\eta^2 - \eta_1^2) \operatorname{sign} \eta.$$

Now it is easy to find, that

$$T^+(\eta)\lambda^-(\eta) - T^-(\eta)\lambda^+(\eta) = [T(\eta) + \lambda(\eta) \operatorname{sign} \eta] \cdot \frac{i\pi}{c} \eta(\eta^2 - \eta_1^2).$$

We enter the integral

$$T_0(z) = \frac{1}{2c} \int_0^1 \frac{\eta^2 - \eta_1^2}{\eta - z} d\eta.$$

and we denote  $T_1(\eta) = T(\eta) + \lambda(\eta) \operatorname{sign} \eta$ . It is clear that  $T_1(\eta)$  is odd function, besides

$$T_1(\eta) = \begin{cases} 1 + 2\eta T_0(-\eta), & \eta > 0, \\ -(1 - 2\eta T_0(\eta)), & \eta < 0. \end{cases}$$

Substituting last two equalities in (7.3), we receive the formula for calculations of coefficient of continuous spectrum

$$E(\eta) = \frac{1}{2c \operatorname{ch}(w_0/\eta)} \left[ \frac{\lambda_1 \eta^2}{\lambda^+(\eta)\lambda^-(\eta)} + \frac{z_0 A_1}{6} \frac{2\eta T_1(\eta) - 3\lambda_\infty \eta^2}{\lambda^+(\eta)\lambda^-(\eta)} \right]. \quad (8.1)$$

Clearly that in complex plane the integral  $T_0(z)$  is calculated under the formula

$$T_0(z) = \frac{1}{2c} \left[ \frac{1}{2} + z + (z^2 - \eta_1^2) \ln \left( 1 - \frac{1}{z} \right) \right],$$

and at  $0 < \eta < 1$  it is calculated by formula

$$T_0(\eta) = \frac{1}{2c} \left[ \frac{1}{2} + \eta + (\eta^2 - \eta_1^2) \ln \left( \frac{1}{\eta} - 1 \right) \right].$$

By means of this function we will present dispersion function in the form

$$\lambda(z) = 1 - zT_0(z) + zT_0(-z),$$

the function  $T(z)$  we express by this integral also

$$T(z) = zT_0(z) + zT_0(-z).$$

The sum of two last expressions is equal

$$\lambda(z) + T(z) = 1 + 2zT_0(-z).$$

We will notice, that the integral  $T(-z)$  is not singul on the cut  $0 < \eta < 1$ . The sum  $\lambda(\eta) + T(\eta) \text{sign } \eta$  on the cut of  $0 < \eta < 1$  it is calculated in an explicit form without quadratures

$$\begin{aligned} \lambda(\eta) + T(\eta) \text{sign } \eta &= 1 + 2\eta T_0(-\eta) = \\ &= 1 + \frac{1}{2c} \left[ \eta - 2\eta^2 + 2\eta(\eta^2 - \eta_1^2) \ln \left( \frac{1}{\eta} + 1 \right) \right]. \end{aligned}$$

Now we consider integral boundary condition (3.6). We rewrite it in the form

$$\int_0^1 \left( \mu^2 - \frac{2}{3}\mu \right) h(-a, \mu) d\mu = -\frac{1 - \alpha_p}{\alpha_p} \cdot \frac{A_1}{36}. \quad (8.2)$$

Here, as it is easy to see,

$$\begin{aligned} h(-a, \mu) &= \frac{E_\infty}{w_0} \mu + \frac{E_0}{w_0} \left[ F(\eta_0, \mu) e^{w_0/\eta_0} + F(-\eta_0, \mu) e^{-w_0/\eta_0} \right] + \\ &+ 2 \int_{-1}^1 F(\eta, \mu) E(\eta) e^{w_0/\eta} d\eta. \end{aligned} \quad (8.3)$$

Let us substitute (7.4) in expansion (7.5) for the function  $h(x, \mu)$ . We receive the following equation

$$\frac{E_\infty}{36} + E_0 m(\eta_0) + 2 \int_{-1}^1 e^{w_0/\eta} m(\eta) E(\eta) d\eta = -\frac{1 - \alpha_p}{\alpha_p} \cdot \frac{w_0 A_1}{36}, \quad (8.4)$$

where

$$m(\eta_0) = e^{w_0/\eta_0} m_0(\eta_0) + e^{-w_0/\eta_0} m_0(-\eta_0).$$

Besides, in the equation (8.4) following designations are accepted

$$m_0(\pm\eta_0) = \int_0^1 \left( \mu^2 - \frac{2}{3}\mu \right) F(\pm\eta_0, \mu) d\mu, \quad m(\eta) = \int_0^1 \left( \mu^2 - \frac{2}{3}\mu \right) F(\eta, \mu) d\mu.$$

Let us substitute in the equation (8.4) the Debaye amplitude  $E_0$ , defined by equation (7.7), and continuous spectrum coefficient  $E(\eta)$ , defined by equality (8.1). We receive the equation

$$\lambda_1 \left( \frac{1}{36\lambda_\infty} - \eta_0 \beta_0 + I_0 \right) + w_0 A_1 \left[ - \left( \frac{1}{3} T(\eta_0) - \frac{\lambda_\infty}{2} \right) \beta_0 + \frac{1}{3} I_1 - \frac{\lambda_\infty}{2} I_0 + \frac{1 - \alpha_p}{36\alpha_p} \right] = 0. \quad (8.5)$$

In the equation (8.5) are entered the following designations

$$\beta_0 = \frac{m_0(\eta_0) e^{w_0/\eta_0} + m_0(-\eta_0) e^{-w_0/\eta_0}}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2) \text{ch}(w_0/\eta_0)},$$



$$\beta(\eta) = \frac{m(\eta)e^{w_0/\eta}}{\lambda^+(\eta)\lambda^-(\eta)\operatorname{ch}(w_0/\eta)}, \quad I_0 = \frac{1}{c} \int_{-1}^1 \eta^2 \beta(\eta) d\eta, \quad I_1 = \frac{1}{c} \int_{-1}^1 \eta T_1(\eta) \beta(\eta) d\eta.$$

Now from equation (8.5) we find the constant  $A_1$

$$z_0 A_1 = - \frac{\lambda_1 \left( \frac{1}{36\lambda_\infty} - \eta_0 \beta_0 + I_0 \right)}{\frac{1}{3} I_1 - \frac{\lambda_\infty}{2} I_0 + \frac{1-\alpha_p}{36\alpha_p} - \frac{\beta_0}{3} T(\eta_0) + \frac{\beta_0}{2} \lambda_\infty \eta_0}. \quad (8.6)$$

The solving of this initial boundary problem on it is finished.

Let us calculate integrals  $m_0(\pm\eta_0)$  and  $m(\eta)$  in an explicit form. Integrals  $m_0(\eta_0)$  and  $m_0(-\eta_0)$  are easily calculated

$$m_0(\eta_0) = (\eta_1^2 - \eta_0^2) \left[ \left( -\frac{1}{6} + \eta_0 \right) + \left( \eta_0^2 - \frac{2}{3}\eta_0 \right) \ln \left( 1 - \frac{1}{\eta_0} \right) \right],$$

$$m_0(-\eta_0) = (\eta_1^2 - \eta_0^2) \left[ \left( -\frac{1}{6} - \eta_0 \right) + \left( \eta_0^2 + \frac{2}{3}\eta_0 \right) \ln \left( 1 + \frac{1}{\eta_0} \right) \right],$$

We calculate the integral  $m(\eta)$

$$m(\eta) = \int_0^1 \left( \mu^2 - \frac{2}{3}\mu \right) \left[ P \frac{\mu\eta - \eta_1^2}{\eta - \mu} - 2c \frac{\lambda(\eta)}{\eta} \delta(\eta - \mu) \right] d\mu =$$

$$= \int_0^1 \frac{(\mu^2 - \frac{2}{3}\mu)(\eta_1^2 - \mu\eta)}{\mu - \eta} d\mu - 2c \left( \eta - \frac{2}{3} \right) \lambda(\eta) \theta_+(\eta).$$

Here  $\theta_+(\eta)$  is the characteristical function of interval  $(0, 1)$ , i.e.

$$\theta_+(\eta) = \begin{cases} 1, & 0 < \eta < 1, \\ 0, & -1 < \eta < 0. \end{cases}$$

From here we receive, that at  $0 < \eta < 1$

$$m(\eta) = (\eta^2 - \eta_1^2) \left( \frac{1}{6} - \eta \right) - (\eta^2 - \eta_1^2) \left( \eta^2 - \frac{2}{3}\eta \right) \ln \left( 1 + \frac{1}{\eta} \right) + 2(\eta^2 - c) \left( \eta - \frac{2}{3} \right),$$

and at  $-1 < \eta < 0$

$$m(\eta) = (\eta^2 - \eta_1^2) \left[ \frac{1}{6} - \eta - \left( \eta^2 - \frac{2}{3}\eta \right) \ln \left( 1 - \frac{1}{\eta} \right) \right].$$

Last two formulas it is possible to unite in one

$$m(\eta) = (\eta^2 - \eta_1^2) \left[ \frac{1}{6} - \eta - \left( \eta^2 - \frac{2}{3}\eta \right) f_+(\eta) \right] + 2(\eta^2 - c) \left( \eta - \frac{2}{3} \right) \theta_+(\eta),$$

where

$$f_+(\eta) = \begin{cases} \ln \frac{1+\eta}{\eta}, & 0 < \eta < 1, \\ \ln \frac{1-\eta}{\eta}, & -1 < \eta < 0. \end{cases}$$

## 10 Energy absorption in layer

The energy of an electromagnetic wave absorbed in the slab of degenerate plasma bi well-known formula [35]

$$Q = \frac{1}{2} \operatorname{Re} \int_{-a}^a j(x) E^*(x) dx. \quad (9.1)$$

Here "asterisk" means complex conjugation, and  $j(x)$  is the density of a current,

$$j(x) = \frac{1}{2} \int_{-1}^1 \mu h(x, \mu) d\mu.$$

In view of one-dimensionality of a problem the equation for electric field has the form  $dE/dx = 4\pi q$ . All quantities have dependence from time  $\exp(-i\omega t)$ , i.e.  $E = E(x) \exp(-i\omega t)$ , and etc. The continuity equation for system a charge – current will be rewrite so:  $dj(x)/dx - i\omega q(x) = 0$ . From last two equations we have

$$i\omega \frac{dE}{dx} = 4\pi \frac{dj(x)}{dx}.$$

Integrating it equality and considering equality to current zero on border, we have

$$j(x) = \frac{i\omega}{4\pi} (E(x) - E_0), \quad (9.2)$$

where  $E_0$  is the amplitude of an external field on border (real quantity).

Substituting (9.2) into (9.1), we find

$$Q = \frac{\omega}{8\pi} \operatorname{Re} \int_{-a}^a i(E(x) - E_0) E^*(x) dx. \quad (9.3)$$

We notice, that  $EE^* = |E|^2 > 0$  is the real quantity, therefore,

$$\begin{aligned} \operatorname{Re} \{i(E - E_0)E^*\} &= -E_0 \operatorname{Re} (iE^*) = \\ &= -E_0 \operatorname{Re} (i(\operatorname{Re} E - i \operatorname{Im} E)) = -E_0 \operatorname{Im} E(x). \end{aligned}$$

Hence, according to (9.3) we have

$$Q = -\frac{\omega E_0}{8\pi} \int_{-a}^a \operatorname{Im} E(x) dx,$$

or, by dimensionless electric field,

$$Q = -\frac{\omega a E_0^2}{8\pi} \operatorname{Im} \int_{-1}^1 e(x_1) dx_1. \quad (9.4)$$

Here the expression received above for electric field is used, which we will present in the form

$$e(x_1) = \frac{\lambda_1}{\lambda_\infty} + 2E_0 \operatorname{ch} \frac{w_0 x_1}{\eta_0} + 2 \int_{-1}^1 \operatorname{ch} \frac{w_0 x_1}{\eta} E(\eta) d\eta. \quad (9.5)$$

Coefficients of continuous and discrete spectra of expansion (9.5) have been calculated in the explicit form.

Let us substitute in the formula (9.4) expression for the electric field (9.5). After integration it is received, that

$$Q = \frac{\omega a_0 E_0^2}{4\pi} Q_0,$$

where  $Q_0$  is the dimensionless part of absorption, defined by formula

$$Q_0 = -\operatorname{Im} Q_1, \quad (9.5')$$

where

$$Q_1 = \frac{\lambda_1}{\lambda_\infty} + 2E_0 \frac{\eta_0}{w_0} \operatorname{sh} \frac{w_0}{\eta_0} + \frac{2}{w_0} \int_{-1}^1 \eta E(\eta) \operatorname{sh} \frac{w_0}{\eta} d\eta. \quad (9.6)$$

The formula (9.5) represents quantity of absorption of electric field energy in the metal layer. It is defined by frequency and amplitude of an external field  $\omega$  and  $E_0$ , width of the layer  $a_0$ , and by parametres  $w_0, z_0, \varepsilon, \Omega$ , reflecting properties metal.

We consider integral from (9.6)

$$J \equiv \int_{-1}^1 \eta E(\eta) \operatorname{sh} \frac{w_0}{\eta} d\eta = \frac{\lambda_1 - w_0 A_1/2}{2c} \int_{-1}^1 \frac{\eta^3 \operatorname{th}(w_0/\eta) d\eta}{\lambda^+(\eta) \lambda^-(\eta)} + \frac{w_0 A_1}{6c} \int_{-1}^1 \frac{\eta^2 \operatorname{th}(w_0/\eta) T_1(\eta) d\eta}{\lambda^+(\eta) \lambda^-(\eta)}.$$

Then taking into account equality

$$\frac{1}{\lambda^+(\eta) \lambda^-(\eta)} = \frac{c}{i\pi\eta(\eta^2 - \eta_1^2)} \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right]$$

we receive

$$J = \left( \lambda_1 - \frac{w_0 A_1}{2} \right) J_1 + \frac{w_0 A_1}{3} J_0,$$

where

$$J_1 = \frac{1}{2\pi i} \int_{-1}^1 \frac{\eta^2 \operatorname{sh}(w_0/\eta)}{(\eta^2 - \eta_1^2) \operatorname{ch}(w_0/\eta)} \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] d\eta,$$

$$J_0 = \frac{1}{2c} \int_{-1}^1 \frac{\eta^2 T_1(\eta) \operatorname{th}(w_0/\eta)}{\lambda^+(\eta) \lambda^-(\eta)} d\eta = \frac{1}{c} \int_0^1 \frac{\eta^2 T_1(\eta) \operatorname{th}(w_0/\eta)}{\lambda^+(\eta) \lambda^-(\eta)} d\eta. \quad (9.7)$$

Let us notice, that integral  $J_1$  it is possible to calculate analytically, and integral  $J_0$  analytically to calculate it is impossible, since function continuation  $T_1(z)$  from the real axis in a complex plane it is carried out by non-analytical function  $|z|(1 + 2zT_0(-|z|))/z$ .

Function zero  $\text{ch}(w_0/z)$  will be necessary for us

$$t_k = \frac{2w_0 i}{\pi(2k+1)}, \quad k = 0, \pm 1, \pm 2, \dots$$

The integral  $J_1$  has been calculated earlier in our work [26]. By means of contour integration and the theory of residues it is found, that

$$J_1 = \left[ \text{Res}_{\infty} + \text{Res}_{\eta_0} + \text{Res}_{-\eta_0} + \text{Res}_{\eta_1} + \text{Res}_{-\eta_1} + \sum_{k=-\infty}^{+\infty} \text{Res}_{t_k} \right] \Phi(z),$$

where

$$\Phi(z) = \frac{z^2 \text{th}(w_0/z)}{\lambda(z)(z^2 - \eta_1^2)}.$$

Calculating all residues from the previous equality, we receive

$$J_1 = -\frac{w_0}{\lambda_{\infty}} + \frac{2\eta_0^2 \text{th}(w_0/\eta_0)}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} + \frac{\eta_1}{\lambda_1} \text{th} \frac{w_0}{\eta_1} - \frac{1}{w_0} \sum_{k=-\infty}^{+\infty} \frac{t_k^4}{\lambda(t_k)(t_k^2 - \eta_1^2)}. \quad (9.8)$$

By means of this expression the formula (9.6) can be presented in the explicit form

$$Q_1 = \frac{\lambda_1}{\lambda_{\infty}} + 2E_0 \frac{\eta_0}{w_0} \text{sh} \frac{w_0}{\eta_0} + \left( \frac{\lambda_1}{w_0} - \lambda_{\infty} \frac{A_1}{2} \right) J_1 + \frac{A_1}{3} J_0. \quad (9.9)$$

In the formula (9.9) Debaye amplitude  $E_0$  is calculated according to (7.7), quantity  $A_1$  is calculated according to (8.6), and integrals  $J_0$  and  $J_1$  are calculated under formulas (9.7) and (9.8) accordingly.

## 11 Conclusions

In the present paper the linearized problem of plasma oscillations in layer (particularly, thin films) in external longitudinal alternating electric field is solved analytically. Specular – accomodative boundary conditions of electron reflection from the plasma boundaries are considered. Coefficients of continuous and discrete spectra of the problem are found, and electron distribution function on the plasma boundary and electric field are expressed in explicit form.

Separation of variables leads to characteristic system of the equations. Its solution in space of the generalized functions allows to find the eigen solutions of initial system of equations of Boltzmann — Vlasov — Maxwell, correspond to continuous spectrum.

Then the discrete spectrum of this problem consisting of zero of the dispersion equations is investigated. Such zeros are three. One zero is infinite remote point of complex plane. It is correspond to the eigen solution "Drude mode". Others two points of discrete spectrum are differing with signs two zero of dispersion function. These zero are correspond to the eigen solution named "Debye mode".

It is found out, that on plane of parameters of a problem  $(\Omega, \varepsilon)$ , where  $\Omega = \omega/\omega_p$ ,  $\varepsilon = \nu/\omega_p$ , there is such domain  $D^+$  (its exterior is domain  $D^-$ ), such, that if a point  $(\Omega, \varepsilon) \in D^-$  Debye mode is absent.

Under eigen solution of initial system its general solution is obtained. By means of boundary conditions in an explicit form expressions for coefficients of discrete and continuous spectra are found. Then in explicit form (without quadratures) absorption quantity of energy of electric field in slab of degenerate plasmas is found.

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